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Phase-space distributions which are probability distributions

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Abstract. The article deals with non-commutative (boson) probability theory. The phase space considered is a linear symplectic space built up from a Hilbert space. Generating functionals for representations of the canonical commutation relations are related to characteristic functionals for random distributions on the underlying Hilbert space. An attempt is made to characterise those Lévy–Khinchin characteristic functionals on phase space which are generating functionals by means of a generalised Heisenberg inequality.

1. Introduction

For a state in quantum mechanics, projection of the Wigner phase-space distribution either onto momentum space or onto configuration space gives the correct probability distribution either for position or for momentum, respectively (Wigner 1932). However, as Wigner remarked (see also Moyal 1949, Hudson 1974) the phase-space distribution itself need not be a true probability distribution. Study of phase-space distributions has led to a non-commutative boson 'probability' theory. Instead of the phase-space distribution we work with its (inverse) Fourier transform, the generating functional. The generating functional for a state of the canonical commutation relations (CCR) plays the role of the characteristic functional, i.e. the Fourier transform of the probability distribution, in commutative probability theory. Skew positive definiteness (Segal 1961) of the boson generating functional is the non-commutative analogue of the positive definiteness of the characteristic functional of a random distribution.

One can define characteristic functionals on a linear phase space (definition 1); the set of generating functionals and the set of characteristic functionals intersect but do not coincide. However, a natural mapping of characteristic functionals to generating functionals gives a one-to-one correspondence between the orthogonal-invariant characteristic functionals and the unitary-invariant generating functionals (proposition 2).

An important class of random processes, including for example white noise and the classical Poisson process, is that of decomposable processes (see Guichardet 1972, appendix E). These have characteristic functionals of the Lévy–Khinchin type and are characterised in theorem 1. Generating functionals of the Lévy–Khinchin type occur in fields of uncoupled oscillators (Araki 1960) and for (continuous) tensor product states for the CCR (Guichardet and Wulfsohn 1970). Characterisations of those Lévy–Khinchin characteristic functionals which are also generating functionals

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are given in propositions 3 and 4 and theorem 2. Fischer (1981) considers similar questions but is concerned with one degree of freedom only and concentrates on extreme points in the set of those characteristic functionals which are also generating functionals.

Lemma 3 was contributed by Professor H Araki.

2. Representations of the canonical commutation relations

Let $\{E, (\,|\,)\}$ be a real Hilbert space. Denote by M_c the Hilbert space $E \oplus iE$ with inner product

$$(z'_1, z'_2) = \tfrac{1}{2}[(x_1|x_2) + (y_1|y_2) + i\{(y_1|x_2) - (x_1|y_2)\}]$$

where $z'_m = 2^{-1/2}(x_m \oplus iy_m)$ and $x_m, y_m \in E$, $m = 1, 2$. Let M denote the Hilbert space $E \oplus E$ with inner product

$$(z_1|z_2)_M = \mathrm{Re}(z'_1, z'_2) = \tfrac{1}{2}\{(x_1|x_2) + (y_1|y_2)\}$$

where $z_m = 2^{-1/2}(x_m \oplus y_m)$. The space M_c is a complexification of M by means of the linear involution $J: x \oplus y \to (-y) \oplus x$ (cf definition 3). Note that M and M_c have the same norm.

A skew-symmetric bilinear form is called *symplectic*. A non-degenerate symplectic form σ can be defined on M by

$$\sigma(z_1, z_2) = -\mathrm{Im}(z'_1, z'_2) = \tfrac{1}{2}\{(x_1|y_2) - (y_1|x_2)\}.$$

A representation of the linear symplectic space (M, σ) in a Hilbert space H is defined to be a mapping W of M to unitary operators of H, strongly continuous on every finite-dimensional subspace and satisfying the commutation relations

$$W(z_1 + z_2) = \exp[i\sigma(z_1, z_2)]W(z_1)W(z_2)$$

for all $z_1, z_2 \in M$. We can replace W by a pair (U, V) of unitary representations of E, continuous on finite-dimensional subspaces and satisfying

$$U(x)V(y) = \exp[-i(x|y)]V(y)U(x).$$

A representation of (M, σ) is also called a *representation of the* CCR.

In a quantum mechanical system, *configuration* (*wavefunction*) *space* can be taken to be a pre-Hilbert space E; its dual space F is referred to as *momentum* (*wavefunction*) *space*. For a formulation of the CCR, *phase space* is identified with the direct sum $E \oplus F$. For our purposes there is no loss in generality in assuming E to be a Hilbert space and identifying F with E. If E is n dimensional one says that the system has n degrees of freedom.

The representation $W_0 = (U_0, V_0)$ of (M, σ) in $L^2(F)$ defined by

$$(U_0(x)\xi)(s) = \exp[-i(x|s)]\xi(s) \qquad (V_0(y)\xi)(s) = \xi(s - y)$$

is called the Schrödinger representation. Because we have chosen our inner products to be linear in the first variable, the U and V of the usual physics notation are interchanged.

Definition 1. (Segal 1961). A complex-valued function Ψ on a vector space H is called positive definite if, for all positive integers n and $x_1, x_2, \ldots, x_n \in H$, the matrices

$[\Psi(x_p - x_q)]_{1\leq p,q\leq n}$ are positive definite. A complex-valued function Ψ on a symplectic space (M, σ) is called skew positive definite if, for all positive integers n and $z_1, z_2, \ldots, z_n \in M$, the matrices $[\exp[i\sigma(z_p, z_q)]\Psi(z_p - z_q)]_{1\leq p,q\leq n}$ are positive definite. A positive-definite function Ψ on H, continuous on each finite-dimensional subspace and satisfying $\Psi(0) = 1$, is called a characteristic functional on H. A skew positive-definite function Φ on (M, σ), continuous on each finite-dimensional subspace and verifying $\Phi(0) = 1$, is called a generating functional for a representation of (M, σ).

3. Results on complete positivity

In this section we relate, for a given state, positivity of an operator-valued matrix and of the matrix whose entries are the expectation values with respect to that state. Although propositions 1(*b*) and (*c*) can be deduced from results of Choi (1972) together with results of Choi and Effros (1977), I have included a proof which is direct and which uses only elementary mathematics.

Let $\boldsymbol{A}$ be an arbitrary C^* algebra. Denote the algebra of $n \times n$ matrices by M_n and n-dimensional Hilbert space by H_n. Let $\mathrm{L}(H)$ and $\mathrm{TC}(H)$ respectively denote the algebras of bounded linear operators and of trace class operators on a Hilbert space H. Consider L and TC in duality, identifying L and TC$'$ with the trace as bilinear form. Denote the Banach dual of $\boldsymbol{A}$ by $\boldsymbol{A}'$, the positive part of $\boldsymbol{A}$ by $\boldsymbol{A}^+$.

For Banach spaces X and Y denote by $\mathrm{L}(X, Y)$ the vector space of bounded linear mappings from X to Y. We identify the vector spaces $M_n(\boldsymbol{A}) \cong \boldsymbol{A} \otimes M_n$ and $\mathrm{L}(M_n, \boldsymbol{A})$ by the correspondence $T([\alpha_{pq}]) = \Sigma_{1\leq p,q\leq n}\, \alpha_{pq} T_{pq}$. We identify $\mathrm{L}(M_n, \boldsymbol{A})$ and $\mathrm{L}(\boldsymbol{A}', M_n)$ so the condition that $T \in M_n(\boldsymbol{A})$ is positive in $\mathrm{L}(\boldsymbol{A}', M_n)$ (i.e. that the matrices $[\omega(T_{pq})]$ are positive definite for all positive linear functionals ω of A) is equivalent to that T is positive in $\mathrm{L}(M_n, \boldsymbol{A})$.

Let $\boldsymbol{D}$ be the cone in $\mathrm{TC}(H \otimes H_n)$ generated by the operators of the form $\rho_1 \otimes \rho_2$, where $\rho_1 \in \mathrm{TC}(H)^+$, $\rho_2 \in M_n^+$. Let $\boldsymbol{C}$ denote the cone $\mathrm{L}(H \otimes H_n)^+$.

Lemma 1. The polar $\boldsymbol{D}^0$ of $\boldsymbol{D}$ can be identified with the set of operators T in $\mathrm{L}(H \otimes H_n)$ such that $[\omega(T_{pq})] \geq 0$ for each positive linear functional ω on $\mathrm{L}(H)$.

Proof. By linearity, $T \in \boldsymbol{D}^0$ if and only if $\mathrm{Tr}((\rho_1 \otimes \rho_2)T) \geq 0$ for all $\rho_1 \in \mathrm{TC}(H)$ and ρ_2 of the form $[\lambda_p \bar{\lambda}_q]$. For $T \in \mathrm{L}(H \otimes H_n)$, we have

$$\mathrm{Tr}((\rho_1 \otimes \rho_2)T) = \mathrm{Tr}\Big(\sum_{p,q} \lambda_p \bar{\lambda}_q \rho_1 T_{pq}\Big) = \sum_{p,q} \lambda_p \bar{\lambda}_q \,\mathrm{Tr}(\rho_1 T_{pq}).$$

Thus $\mathrm{Tr}((\rho_1 \otimes \rho_2)T) \geq 0$ if and only if $[\mathrm{Tr}(\rho_1 T_{pq})] \geq 0$. Identifying $\mathrm{TC}(H)^+$ with the positive linear functionals of $\mathrm{L}(H)$ the lemma follows.

Proposition 1. Let H and H_n be Hilbert spaces, $\dim H_n = n > 1$, $\dim H > 1$. (*a*) Let $\boldsymbol{A}$ be a commutative C^* algebra on a real or complex Hilbert space K and let $T \in M_n(\boldsymbol{A})$, $n > 1$. If for all vector states ω of $\boldsymbol{A}$ the matrices $[\omega(T_{pq})]$ are positive definite, then $T \geq 0$ in $M_n(\boldsymbol{A})$. (*b*) There exists $T \in \mathrm{L}(H \otimes H_n)$, $T \notin \mathrm{L}(H \otimes H_n)^+$, such that $[\omega(T_{pq})] \geq 0$ for all positive linear functionals ω on $\mathrm{L}(H)$. (*c*) Let $\boldsymbol{A} = \mathrm{L}(H)$. If $T \geq 0$ in $M_n(\boldsymbol{A})$ then $T \geq 0$ in $\mathrm{L}(\boldsymbol{A}', M_n)$, but the converse does not always hold.

Proof. (*a*) We may assume that $K = L^2(K, B, \mu)$ and $\boldsymbol{A} \subset L^\infty(X, B, \mu)$ for some finite-measure space (X, B, μ). Since $(L^1)^+ \subset \boldsymbol{A}^+$, by hypothesis $[\int_X f(x) T_{pq}(x)\, \mathrm{d}\mu(x)] \geq 0$ for all $f \in (L^1)^+$. Thus $[T_{pq}(x)] \geq 0$ almost everywhere with respect to μ. Writing $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in \bigoplus_{i=1}^n K_i$ where each $K_i = K$

$$(T\psi|\psi) = \sum_{p,q} \int_X \bar{\psi}_q(x) T_{pq}(x) \psi_p(x)\, \mathrm{d}\mu(x) = \int_X ([T_{pq}(x)]\psi(x)|\psi(x))\, \mathrm{d}\mu(x) \geq 0$$

so $T \geq 0$.

(b) Let $n = 2$, $\dim H = 2$. Choose T of the form

$$\begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -\lambda \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}.$$

The leading minors are all positive and $\det T = 1 - \lambda^2$. Any positive linear functional ω on M_2 may be represented by a non-negative matrix $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ with $a \geq 0$, $a + d \geq 0$ and $ad - bc \geq 0$. So $[\omega(T_{pq})] = [\begin{smallmatrix} a+d & -\lambda c \\ -\lambda b & a+d \end{smallmatrix}]$ is positive definite when $\lambda \leq \sqrt{2}$. Thus taking $\lambda \in [1, \sqrt{2}]$, the theorem is true for the above case. In general, for $\dim H > 1$, $n > 1$ the above T can be identified with an element of $\mathrm{L}(H \otimes H_n)$.

(c) Obviously $\boldsymbol{C}^0 \supset \boldsymbol{D}$, so $\boldsymbol{D}^0 \supset \boldsymbol{C}^{00} \supset \boldsymbol{C}$ but by (b), $\boldsymbol{C} \neq \boldsymbol{D}^0$.

We relate the above results to the more general concept of complete positivity. Complete positivity can be defined for linear mappings between matrix-ordered spaces, as defined by Choi and Effros (1977); we are concerned only with $\boldsymbol{A}$ and $\boldsymbol{A}'$, both known to be matrix ordered. Given matrix-ordered spaces V and W, a linear mapping $\varphi: V \to W$ is *completely positive* if for each $n \in \mathbb{N}$ the mapping $\varphi_n: M_n(V) \to M_n(W)$, defined by $\varphi_n([v_{pq}]) = [\varphi(v_{pq})]$, is positive. By lemma 4.3 of Choi and Effros (1977), $T \geq 0$ in $M_n(\boldsymbol{A})$ if and only if T is completely positive in $\mathrm{L}(\boldsymbol{A}', M_n)$. Thus proposition 1($a$) shows that, in the commutative case, positivity and complete positivity are equivalent for the mapping $\omega \to [\omega(T_{pq})]$ from $\boldsymbol{A}'$ to M_n. This generalises Stinespring's (1955) result on continuous linear functionals.

4. Characteristic functionals and generating functionals

Positive-definite functions are not necessarily skew positive definite; an example is the function Ψ_0 identically 1. We describe below a class of skew positive-definite functions which are not positive definite.

Consider the Schrödinger representation of (M, σ) for finite-dimensional $M = \mathbb{R}^{2m}$. Take L^2 relative to Lebesgue measure. The *Hermite functions*, defined for $n \in \mathbb{N}$ by

$$\zeta_n(s) = (2^n n!\, \sqrt{\pi})^{-1/2} \exp(-\tfrac{1}{2}s^2) H_n(s)$$

where H_n denotes the nth Hermite polynomial, form an orthonormal basis for $L^2(\mathbb{R})$. The Fourier transform $\hat{\Phi}_n$ of the generating functional Φ_n, where $\Phi_n(z) = (W_0(z)\zeta_n|\zeta_n)$, is called the *phase-space distribution* of the stationary state (vector) ζ_n. None of the Φ_n, excepting Φ_0, are positive definite; indeed the Φ_n, and therefore also the $\hat{\Phi}_n$, are mutually orthogonal in L^2 and $\hat{\Phi}_0(w) = \exp(-\frac{1}{2}\|w\|^2) \geq 0$ so the other $\hat{\Phi}_n$ are negative on sets of positive measure. An example is $\hat{\Phi}_1(w) = (4\|w\|^2 - 1)\exp(-2\|w\|^2)$, negative whenever $\|w\|<\frac{1}{2}$. For m degrees of freedom one can similarly define $\Phi_{\boldsymbol{n}}$, $\boldsymbol{n}=(n_1, n_2, \ldots, n_m)$, from analogous Hermite functions in $L^2(\mathbb{R}^m)$. The $\Phi_{\boldsymbol{n}}$, $\boldsymbol{n}\neq 0$, can be extended canonically to generating functionals, not positive definite, on any linear phase space.

The following lemma is well known (cf Schur 1911).

Lemma 2. If $[a_{pq}]$ and $[b_{pq}]$ are positive-definite matrices, then the matrices $[a_{pq}b_{pq}]$ and $[\exp(a_{pq})]$ are also positive definite.

Let Φ be a generating functional. Since

$$\Phi(z'_p - z'_q)\exp\{-\tfrac{1}{2}\|z'_p - z'_q\|^2\}$$
$$= \Phi(z'_p - z'_q)\exp\{-\mathrm{i}\,\mathrm{Im}(z'_p, z'_q)\}\times\exp\{-\tfrac{1}{2}(\|z'_p\|^2+\|z'_q\|^2)\}\times\exp\{(z'_p, z'_q)\}$$

each factor the (p, q)th entry of a positive-definite matrix, the operation of multiplication by $\exp(-\frac{1}{2}\|z\|^2)$ maps generating functionals to characteristic functionals; this exhibits the well known fact that *normal-ordered phase-space distributions are non-negative.* A similar normalising mapping occurs in the following proposition.

We associate with (M, σ) a von Neumann algebra $\boldsymbol{A}_{M,\sigma}$ such that there is a bijection between its normal states and the set of generating functionals for (M, σ) (see Guichardet 1968).

Proposition 2. Let $\mathrm{U}(M, \sigma)$ denote the group of automorphisms of $\boldsymbol{A}_{M,\sigma}$ induced by the unitary group of M_c. The operation P, defined by $(P\Psi)(z)=\exp(-\frac{1}{2}\|z\|^2)\Psi(z)$, maps characteristic functionals on M to generating functionals on (M, σ) and gives rise to a bijection of the set of orthogonal-invariant characteristic functionals on M onto the set of normal states of $\boldsymbol{A}_{M,\sigma}$ invariant for $\mathrm{U}(M, \sigma)$.

Proof. The first statement holds since the product of a skew positive-definite function by a positive-definite function is skew positive definite. As in Umemura (1950), the characteristic functionals invariant under the orthogonal group of M can be shown to be of the form $\int_0^\infty \exp(-\frac{1}{2}\lambda\|z\|^2)\,\mathrm{d}\boldsymbol{m}(\lambda)$ where $\boldsymbol{m}$ denotes a probability measure on $[0, +\infty)$, and as in Segal (1962), the states of $\boldsymbol{A}_{M,\sigma}$ invariant under $\mathrm{U}(M, \sigma)$ can be shown to be of the form $\int_1^\infty \exp(-\frac{1}{2}\lambda\|z\|^2)\,\mathrm{d}\boldsymbol{n}(\lambda)$ where $\boldsymbol{n}$ denotes a probability measure on $[1, +\infty)$; the proposition follows.

Definition 2. A function on a real Hilbert space V is called a Lévy–Khinchin function if it is of the form

$$\psi_{A,u,\nu}: x \to \exp\left(\mathrm{i}(u|x) - \tfrac{1}{2}(Ax|x) + \int_V K(v, x)\,\mathrm{d}\nu(v)\right)$$

where A is a symmetric operator on V,

$$K(v, x) = \exp[\mathrm{i}(v|x)] - 1 - \mathrm{i}(v|x)(1+\|v\|^2)^{-1}$$

and ν is a σ-finite measure on the Borel sets of V satisfying

$$\nu(\{0\}) = 1 \qquad \text{and} \qquad \int_V \|v\|^2(1+\|v\|^2)^{-1}\,\mathrm{d}\nu(v) < \infty.$$

We shall write Ψ_A to denote $\Psi_{A,0,0}$.

Lemma 3.

$$\lim_{\lambda\to\infty} \lambda^{-2} \int_V K(v, \lambda x)\, d\nu(v) = 0.$$

Proof. Let

$$I = \int_V \|v\|^2(1+\|v\|^2)^{-1}\, d\nu(v) < \infty.$$

For $1 \geq a \geq 0$, we define

$$I_1(a) = \int_{\|v\|\geq a} d\nu(v) \leq (1+a^2)a^{-2} \leq 2Ia^{-2}$$

$$I_2(a) = \int_{\|v\|\leq a} \|v\|^2\, d\nu(v) \to 0 \qquad \text{as} \quad a \to 0.$$

For $K(v, \lambda x)$ we have the following two estimates:

$$|K(v, \lambda x)| \leq 2 + \tfrac{1}{2}\|\lambda x\|$$

$$\begin{aligned}|K(v, \lambda x)| &\leq |\exp[i(v|\lambda x)] - 1 - i(v|\lambda x)| + |(v|\lambda x)|\,\|v\|^2(1+\|v\|^2)^{-1}\\ &< \tfrac{1}{2}(|(v|\lambda x)|^2 + \|v\|^2\|\lambda x\|)\\ &< \tfrac{1}{2}\|v\|^2(\|\lambda x\|^2 + \|\lambda x\|).\end{aligned}$$

Hence

$$\left|\int_V K(v, \lambda x)\, d\nu(v)\right| \leq a^{-2}I(4+\|\lambda x\|) + \tfrac{1}{2}I_2(a)(\|\lambda x\|^2 + \|\lambda x\|).$$

For any given $\varepsilon > 0$, we choose $a_\varepsilon > 0$ such that

$$I_2(a_\varepsilon)(\|x\|^2 + \|x\|) < \varepsilon.$$

For this a_ε, we choose $\Lambda_\varepsilon > 1$ such that, for all $|\lambda| > \Lambda_\varepsilon$,

$$a_\varepsilon^{-2}(4+\|\lambda x\|)I \leq \tfrac{1}{2}\lambda^2\varepsilon.$$

We then have

$$\lambda^{-2}\left|\int_V K(v, \lambda x)\, d\nu(v)\right| < \varepsilon \qquad \text{for} \qquad |\lambda| > \Lambda_\varepsilon.$$

Theorem 1. The Lévy–Khinchin function $\Psi_{A,u,\nu}$ is positive definite if and only if $A \geq 0$.

Proof. The function $\Psi_{0,u,\nu}$ is positive definite (see Parthasarathy 1967). It follows from lemma 2 that $\Psi_{A,u,\nu}$ also is positive definite. To prove the converse it is sufficient to find y satisfying

$$2(1 - \mathrm{Re}\, e^{i\theta}\Psi(y)) = \Psi(0-0) + \Psi(y-y) - e^{i\theta}\Psi(y-0) - e^{-i\theta}\Psi(0-y) < 0$$

for some real θ, where $\Psi = \Psi_{A,u,\nu}$. If $A \geq 0$ does not hold there exists x such that $(Ax|x) < 0$. Then

$$|\Psi(\lambda x)| = \exp\left(\lambda^2\left[-\tfrac{1}{2}(Ax|x) + \lambda^{-2}\,\mathrm{Re}\int_V K(v, \lambda x)\, d\nu(v)\right]\right).$$

By lemma 3, there exists $\Lambda > 1$ such that, for $\lambda > \Lambda$,

$$-\tfrac{1}{2}(Ax|x)+\lambda^{-2}\,\mathrm{Re}\int_V K(v,\lambda x)\,\mathrm{d}\nu(v) > -\tfrac{1}{4}(Ax|x) > 0.$$

For any one such λ, choose θ so that $e^{i\theta}\Psi(\lambda x) = |\Psi(\lambda x)|$. Then we have $\mathrm{Re}\, e^{i\theta}\Psi(\lambda x) > 1$.

Lemma 4. Let A be a Hermitian operator on a (real or complex) Hilbert space H of dimension greater than one. A necessary and sufficient condition that for all positive integers n and $z_1, z_2, \ldots, z_n \in H$ the matrices $[\exp\{(Az_p|z_q)\}]_{1\leqslant p,q\leqslant n}$ be positive definite is that $A \geqslant 0$.

Proof. The sufficiency is evident from lemma 2. To prove the necessity we suppose that $A \not\geqslant 0$ and find a version of the matrix which is not positive definite. Choose z_1 and z_2 linearly independent and such that $(A(z_1-z_2)|z_1-z_2)<0$. Thus $(Az_1|z_1)+(Az_2|z_2)<(Az_1|z_2)+(Az_2|z_1)$. By exponentiating this inequality it is easily seen that the determinant of the matrix $[\exp\{(Az_p|z_q)\}]_{1\leqslant p,q\leqslant 2}$ is negative.

Definition 3. We define a transform J on M_c by $J(x \oplus iy) = (-y) \oplus ix$. It is anti-unitary, $V(Jy) = V(-y)$ and $U(Jx) = U(x)$. Thus J is the operator, defined by Araki (1960), describing a *reversal of motion at fixed time*. We denote the restriction of J to M also by J. By identifying $\mathrm{L}(M_c)$ with $\mathrm{L}(E) \otimes M_2(\mathbb{C})$ we may express $T \in \mathrm{L}(M_c)$ as a 2×2 matrix $[T_{pq}]$ with entries in $\mathrm{L}(E)$. We extend $S \in \mathrm{L}(M)$ to $\mathrm{L}(M_c)$ and denote $S + \frac{1}{2}iJ$ by S^c. If

$$S = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$$

then

$$S^c = \begin{bmatrix} A & C - \frac{1}{2}iI \\ C^* + \frac{1}{2}iI & B \end{bmatrix}.$$

Note that if $T \geqslant 0$ on M_c then also $T \geqslant 0$ on M.

Proposition 3. A Lévy–Khinchin characteristic functional $\Psi_{S,u,\nu}$ on M is skew positive definite if and only if $S^c \geqslant 0$ on M.

Proof. For any positive integer n let $z_1, z_2, \ldots, z_n \in M$. For $z_p, z_q \in M$,

$$\Psi_{S,u,\nu}(z_p - z_q)\exp[i\sigma(z_p, z_q)]$$
$$= \exp\{-\tfrac{1}{2}(Sz_p|z_p)_M + \tfrac{1}{2}(Sz_q|z_q)_M\} \times \Psi_{0,u,\nu}(z_p - z_q) \times \exp\{(S^c z_p|z_q)_M\}.$$

By lemma 2, if $S^c \geqslant 0$ then $\Psi_{S,u,\nu}$ is skew positive definite. The function Φ defined by

$$\Phi(z_p - z_q) = \exp\{\tfrac{1}{2}[(Sz_p|z_p)_M + (Sz_q|z_q)_M]\}$$

is positive definite since

$$\sum_{p,q} \Phi(z_p - z_q)c_p\bar{c}_q = \left|\sum_p \exp\{\tfrac{1}{2}(Sz_p|z_p)_M\}c_p\right|^2.$$

Write

$$\exp\left\{i(u|x) + \int_{\|\nu\|\leqslant m} K(v,x)\,\mathrm{d}\nu(v)\right\} \qquad m \in \mathbb{N}$$

as $\Psi_{u,\nu,n}(x)$. If $\Psi_{S,u,\nu}$ is skew positive definite then

$$[\Phi\Psi_{-u,\nu,n}\Psi_{S,u,\nu}(z_p - z_q)\exp\{i\sigma(z_p, z_q)\}]_{1\leqslant p,q\leqslant n}$$

is positive definite. So, letting $m\to\infty$, also $[\exp\{(S^c z_p|z_q)_M\}]_{1\leqslant p,q\leqslant n}$ will be positive definite. By lemma 4, $S^c\geqslant 0$.

Definition 4. For a given $\Psi_{S,u,\nu}$, we call the relation $[(S^c_{pq}x|x)]\geqslant 0$ for all $x\in E$, i.e.

$$(Ax|x)(Bx|x)-(Cx|x)(C^*x|x)-\tfrac{1}{4}(x|x)^2\geqslant 0 \qquad \text{for all} \quad x\in E, A\geqslant 0$$

the *Heisenberg inequality.*

We know from proposition 1 that the Heisenberg inequality need not imply that $S^c\geqslant 0$ on M but that, if the $S_{p,q}$ are commuting operators, then the Heisenberg inequality implies that $S^c\geqslant 0$. Thus, for one degree of freedom, the Heisenberg inequality is equivalent to the condition that $S^c\geqslant 0$ on M, and so, as is well known (see Araki 1960), for one degree of freedom the Heisenberg inequality is necessary and sufficient for $\Psi_{S,u,\nu}$ to be a generating functional.

Proposition 4. In order that a Lévy–Khinchin characteristic functional $\Psi_{S,u,\nu}$ on M be a generating functional for (M,σ) it is sufficient, but not generally necessary, that $S^c\geqslant 0$ on M_c.

Proof. If $S^c\geqslant 0$ on M_c then also $S^c\geqslant 0$ on M and, by proposition 3, $\Psi_{S,u,\nu}$ is a generating functional; on the other hand, were $S^c\geqslant 0$ a necessary condition, $S^c\geqslant 0$ on M would imply $S^c\geqslant 0$ on M_c, not generally true.

Theorem 2. Given a Lévy–Khinchin characteristic functional on M, if it is a generating functional for (M,σ) then the Heisenberg inequality is satisfied. When the number of degrees of freedom is greater than one the Heisenberg inequality is not generally sufficient for the characteristic functional to be a generating functional.

Proof. Suppose that $\psi_{S,u,\nu}$ is a generating functional. Let $z_1=0$, $z_2=\lambda x\oplus 0$ and $z_3=0\oplus\lambda x$, where $x\in E$, $\lambda\in\mathbb{R}$. Letting λ be small enough that one may neglect powers of λ higher than 4, the determinant of the 3×3 matrix $[\psi_{S,u,\nu}(z_p - z_q)\exp\{i\sigma(z_p, z_q)\}]$ can be seen to be

$$\lambda^4\exp\Big((Ax|x)+2\int[1-\cos(v|x)]\,d\nu(v)\Big)[(Ax|x)(Bx|x)-(Cx|x)(C^*x|x)-\tfrac{1}{4}(x|x)^2].$$

Since $\Psi_{S,u,\nu}$ is a characteristic functional, $S\geqslant 0$ by theorem 2. So also $A\geqslant 0$ and, since the above mentioned determinant is non-negative, the Heisenberg inequality is satisfied. To prove the second statement of the theorem we find an $S\geqslant 0$ such that S^c is not non-negative and the Heisenberg inequality holds. For two degrees of freedom let

$$S=\begin{bmatrix}1 & \varepsilon & \lambda & 0\\ \varepsilon & 1 & 0 & \lambda\\ \lambda & 0 & 1 & 0\\ 0 & \lambda & 0 & 1\end{bmatrix}$$

and write $x = \binom{x_1}{x_2}$. For $x_2 \neq 0$ and $X = x_1 x_2^{-1}$

$$\det[(S^c_{pq}x|x)] = aX^4 + 4bX^3 + 6cX^2 + 4bX + a$$

where $a = (\frac{3}{4} - \lambda^2)$, $b = \frac{1}{2}\varepsilon$ and $c = \frac{1}{3}a$. When $x_2 = 0$ then $\det[(S^c_{pq}x|x)] = ax_1^4$. Thus if $a > 0$ and the biquadratic has no real roots then the Heisenberg inequality is satisfied. This is so if $ac - b^2 > 0$ and $h = a^2c + 2b^2c - ab^2 - c^3 > 0$ (Burnside and Panton 1912). Choosing $\lambda = 0.8$, $\varepsilon = 0.12$, then $S \geqslant 0$, $a = 0.11$, $ac - b^2 > 0.004$, $h > 0.0002$, while S^c is not non-negative, having determinant $\frac{9}{16} - \frac{3}{2}\lambda^2 + \lambda^4 - \varepsilon^2 < -0.002$.

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